Convergence of Sequences and Series

Complex Series, Limsup/Liminf, and Uniform Convergence

Lecture Notes

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1 Crash Course in Complex Sequences and Series

Nothing here is new – it is all reducible to the real case via real and imaginary parts.

Some details.

$$z_n \to z \iff (\operatorname{Re} z_n \to \operatorname{Re} z \text{ and } \operatorname{Im} z_n \to \operatorname{Im} z) \iff |z_n - z| \to 0.$$

$$\sum_{n=1}^{\infty} z_n \text{ converges} \iff \left\{\sum \operatorname{Re} z_n \text{ converges and } \sum \operatorname{Im} z_n \text{ converges}\right\}.$$

Cauchy criterion for series

$\sum z_n$ converges if and only if

$$\forall \varepsilon > 0 \ \exists N \ \forall m > k > N : \left| \sum_{n=k}^{m} z_n \right| < \varepsilon.$$

Remark. $\sum_{n=1}^{\infty} z_n$ converges $\implies |z_n| \to 0$ (take $m = k$ in the Cauchy criterion).

Definition (absolute convergence)

$\sum z_n$ converges **absolutely** if $\sum |z_n| < \infty$.

Fact (absolute convergence $\iff$ bounded partial sums). $\sum z_n$ converges absolutely if and only if the partial sums $\sum_{n=0}^{N} |z_n|$ are bounded (as $N \to \infty$): since $|z_n| \geq 0$, these partial sums are non-decreasing, so (by the monotone convergence of real sequences) they converge – i.e. $\sum |z_n| < \infty$ – exactly when they are bounded above.

Important fact. $\sum |z_n|$ converges $\implies \sum z_n$ converges (apply the Cauchy criterion to $\sum |z_n|$ and the triangle inequality $|\sum_{n=k}^{m} z_n| \leq \sum_{n=k}^{m} |z_n|$).
Another important fact. If $\sum z_n$ converges, then $\lim_{k\to\infty} \sum_{n=k}^{\infty} z_n = 0$ (the tails of a convergent series vanish): apply the Cauchy criterion with $m \to \infty$.

2 Interlude: Supremum, Infimum, Limsup, and Liminf

We recall these notions for real sequences, since they will be needed for the radius of convergence of power series.

Definition (supremum and infimum)

Let $A \subseteq \mathbb{R}$. We say $a \in \mathbb{R}$ is an **upper bound** for A if $x \in A \implies x \le a$; and $a := \sup A$ if a is an upper bound and every other upper bound a' satisfies $a' \ge a$ (the *least* upper bound). If A has no upper bound, $\sup A := +\infty$.
Symmetrically, a is a **lower bound** for A if $x \in A \implies x \ge a$; and $a := \inf A$ if a is a lower bound and every other lower bound a' satisfies $a' \le a$ (the *greatest* lower bound). If A has no lower bound, $\inf A := -\infty$.

Example. $A = \{(-1)^n : n \in \mathbb{N}\} = \{-1, 1\}$ – lower bounds are any $a \le -1$, upper bounds are any $a \ge 1$.

Definition (limsup and liminf)

Let (a_n) be a real sequence. For each n let $R_n := \{a_k : k \ge n\}$, $u_n := \sup R_n$, $\ell_n := \inf R_n$. Since $R_1 \supseteq R_2 \supseteq \cdots$, the sequence (u_n) is non-increasing and (ℓ_n) is non-decreasing; moreover $\ell_1 \le \ell_n \le u_n \le u_1$ for every n. So (u_n) is monotone and bounded below (by ℓ_1), and (ℓ_n) is monotone and bounded above (by u_1); hence both converge, and we define

$$\limsup_{n\to\infty} a_n := \lim_n u_n = \inf_n u_n, \qquad \liminf_{n\to\infty} a_n := \lim_n \ell_n = \sup_n \ell_n.$$

Equivalently,

$$\limsup_n a_n = \sup\{b : \#\{n : a_n > b\} \text{ is infinite}\} = \inf\{c : \#\{n : a_n > c\} \text{ is finite}\}.$$

Remark (unbounded sequences). If (a_n) is unbounded above, then every tail $R_n = \{a_k : k \ge n\}$ is also unbounded above (removing the finitely many terms with index $< n$ cannot bound it), so $u_n = \sup R_n = +\infty$ for *every* n, and hence $\limsup_n a_n = +\infty$. Likewise, if (a_n) is unbounded below, then $\ell_n = \inf R_n = -\infty$ for every n, so $\liminf_n a_n = -\infty$. These two phenomena are independent: e.g. $a_n = (-1)^n n$ is unbounded both above and below, with $\limsup a_n = +\infty$ and $\liminf a_n = -\infty$; while $a_n = n$ for n even, $a_n = 0$ for n odd, is unbounded above but bounded below, with $\limsup a_n = +\infty$ but $\liminf a_n = 0$ (finite) – note that being bounded below is *not* enough to force a finite $\liminf$: it is the odd terms, returning to 0 infinitely often, that keep $\ell_n = 0$ for every n; a monotone sequence like $a_n = n$, by contrast, has $\liminf a_n = \limsup a_n = \lim a_n = +\infty$ as well (by the Fact below).

Example. $a_n = (-1)^n + \dfrac{\sin n}{n}$. Then $\limsup a_n = 1$ and $\liminf a_n = -1$: for $b < 1$, $\{n : a_n > b\}$ is infinite (it contains all sufficiently large even n, since $a_n = 1 + \sin(n)/n \to 1$ along evens), while for $b > 1$ only finitely many n satisfy $a_n > b$.

Property (characterization of lim sup)

$\limsup_n a_n$ is the unique number ℓ such that:

(1) $\forall \varepsilon > 0$: $\{n : a_n > \ell + \varepsilon\}$ is finite;

(2) $\forall \varepsilon > 0$: $\{n : a_n > \ell - \varepsilon\}$ is infinite.

Proof. Let $\ell := \limsup_n a_n = \inf_n u_n$, where $u_n = \sup R_n \downarrow \ell$.

(1) holds for ℓ. Since $u_n \downarrow \ell$, given $\varepsilon > 0$ there is N with $u_N < \ell + \varepsilon$. Then $a_k \le u_N < \ell + \varepsilon$ for every $k \ge N$, so $\{n : a_n > \ell + \varepsilon\} \subseteq \{1, \dots, N-1\}$ is finite.

(2) holds for ℓ. Given $\varepsilon > 0$, suppose toward a contradiction that $\{n : a_n > \ell - \varepsilon\}$ is finite; then there is N with $a_k \le \ell - \varepsilon$ for all $k \ge N$, so $u_N = \sup_{k \ge N} a_k \le \ell - \varepsilon < \ell$, contradicting $u_N \ge \inf_n u_n = \ell$. So $\{n : a_n > \ell - \varepsilon\}$ is infinite.

Uniqueness. Suppose ℓ' also satisfies (1) and (2). If $\ell' > \ell$, let $\varepsilon := (\ell' - \ell)/2 > 0$; then $\ell + \varepsilon = \ell' - \varepsilon$, and (1) for ℓ says $\{n : a_n > \ell + \varepsilon\}$ is finite, while (2) for ℓ' says $\{n : a_n > \ell' - \varepsilon\} = \{n : a_n > \ell + \varepsilon\}$ is infinite – a contradiction. If $\ell' < \ell$, let $\varepsilon := (\ell - \ell')/2 > 0$; then $\ell' + \varepsilon = \ell - \varepsilon$, and (1) for ℓ' says $\{n : a_n > \ell' + \varepsilon\}$ is finite, while (2) for ℓ says $\{n : a_n > \ell - \varepsilon\} = \{n : a_n > \ell' + \varepsilon\}$ is infinite – again a contradiction. So $\ell' = \ell$. □

Another characterization (of $\liminf$). By the symmetric argument (or applying the above to $-a_n$),

$$\liminf_n a_n = \inf\{b : \#\{n : a_n < b\} \text{ infinite}\} = \sup\{c : \#\{n : a_n < c\} \text{ finite}\},$$

and $\liminf_n a_n$ is likewise the unique number ℓ with $\{n : a_n < \ell - \varepsilon\}$ finite and $\{n : a_n < \ell + \varepsilon\}$ infinite for every $\varepsilon > 0$; the proof is the same, applied to $-a_n$.

Fact

$\lim_n a_n$ exists (possibly $\pm\infty$) if and only if $\limsup_n a_n = \liminf_n a_n$, in which case both equal $\lim_n a_n$.

Proof. Since $a_n \in R_n$, we have $\ell_n \le a_n \le u_n$ for every n.

($\Leftarrow$) If $\limsup_n a_n = \liminf_n a_n =: L$, then $u_n \to L$ and $\ell_n \to L$, so $\ell_n \le a_n \le u_n$ squeezes $a_n \to L$.

($\Rightarrow$) Suppose $a_n \to L$.

Case L finite. Fix $\varepsilon > 0$. Eventually $a_n < L + \varepsilon$, so for N large enough $u_N = \sup_{k \ge N} a_k \le L + \varepsilon$, giving $\limsup_n a_n = \inf_n u_n \le L + \varepsilon$. Likewise eventually $a_n > L - \varepsilon$, so $\ell_N \ge L - \varepsilon$ for N large, giving $\liminf_n a_n = \sup_n \ell_n \ge L - \varepsilon$. As $\varepsilon > 0$ was arbitrary, $\limsup_n a_n \le L \le \liminf_n a_n$; since always $\ell_n \le u_n$ forces $\liminf_n a_n \le \limsup_n a_n$, both inequalities are equalities and $\limsup_n a_n = \liminf_n a_n = L$.

Case $L = +\infty$. For every M, eventually $a_n > M$, so $\ell_N = \inf_{k \ge N} a_k \ge M$ for N large, giving $\liminf_n a_n = \sup_n \ell_n \ge M$; as M was arbitrary, $\liminf_n a_n = +\infty$, and then $\limsup_n a_n \ge \liminf_n a_n = +\infty$ forces $\limsup_n a_n = +\infty$ as well. The case $L = -\infty$ is symmetric (apply the above to $-a_n$). □

3 Pointwise versus Uniform Convergence

Let $f_n, f : K \to \mathbb{C}$, $K \subseteq \mathbb{C}$.

Definition (pointwise convergence)

$f_n \to f$ **pointwise** on K if

$$\forall z \in K \ \forall \varepsilon > 0 \ \exists N(z, \varepsilon) : n > N(z, \varepsilon) \implies |f_n(z) - f(z)| < \varepsilon,$$

i.e., $|f_n(z) - f(z)| \to 0$ for each $z \in K$ separately.

Definition (uniform convergence)

$f_n \to f$ **uniformly** on K, written $f_n \rightrightarrows f$, if

$$\forall \varepsilon > 0 \; \exists N(\varepsilon) : n > N(\varepsilon) \implies |f_n(z) - f(z)| < \varepsilon \quad \forall z \in K,$$

equivalently $\sup_{z \in K} |f_n(z) - f(z)| \to 0$.

Theorem

A uniform limit of a sequence of continuous functions is continuous.

Proof. Same proof as in the real case: fix $z_0 \in K$ and $\varepsilon > 0$; choose n with $\sup_{z \in K} |f_n(z) - f(z)| < \varepsilon/3$, then a neighbourhood of z_0 on which $|f_n(z) - f_n(z_0)| < \varepsilon/3$ (continuity of f_n); the triangle inequality gives $|f(z) - f(z_0)| < \varepsilon$ there. □

Example (pointwise but not uniform). On $[0,1]$, $x^n \to \begin{cases} 0, & x < 1 \\ 1, & x = 1 \end{cases}$ pointwise, but the limit is discontinuous, so the convergence cannot be uniform.

4 The Weierstrass M-Test

Theorem (M-test for uniform convergence of series)

Let $K \subseteq \mathbb{C}$ and suppose $\sum_{n=1}^{\infty} M_n < \infty$ with $M_n \geq 0$. If $\forall z \in K \; \forall n$, $|f_n(z)| \leq M_n$, then $\sum f_n(z)$ converges uniformly on K.

Proof. Since $\sum M_n$ converges, $\forall \varepsilon > 0 \; \exists N$ such that $m > n > N \implies \sum_{k=n}^{m} M_k < \varepsilon$. Then for such m, n and any $z \in K$,

$$\left| \sum_{k=n}^{m} f_k(z) \right| \leq \sum_{k=n}^{m} |f_k(z)| \leq \sum_{k=n}^{m} M_k < \varepsilon,$$

so by the Cauchy criterion (uniformly in z), $\sum_{k=1}^{\infty} f_k(z) =: f(z)$ exists for every $z \in K$. Moreover

$$\left| f(z) - \sum_{k=1}^{n} f_k(z) \right| \leq \sum_{k=n+1}^{\infty} M_k \to 0,$$

and the bound is independent of z, so the convergence is uniform. □